The Glove Problem

Spring 2022 ARML Power Contest

The Background

In 1977, the puzzle master Martin Gardner published a puzzle in *Isaac Asimov's Science Fiction Magazine* about doctors who need to operate on patients while trying to avoid the Barsoomian flu. The only problem is that they do not have enough gloves for each doctor-patient encounter to use a pair of sterile gloves. This Power Contest will explore the puzzle and some generalizations. (The word "Barsoomian" was a nod to another science fiction author, Edgar Rice Burroughs.)

In the original version of the problem, there were three doctors and one patient. Each doctor had to perform an operation on the patient, but there were only two pairs of surgical gloves available. Since the flu transferred by touch, no doctor or patient could be exposed to a surface of a glove that had previously been exposed to a different person. Their solution: Doctor X wears *both* pairs of gloves and operates. Doctor Y then wears just the pair of gloves that touched the patient—the inside of those gloves are still safe because Doctor X wasn't touching them! Doctor Z then takes the gloves that were worn by Doctor X, *turns them inside-out* so as to wear them while touching the remaining safe side, and then places the (now out)side, possibly infected by Doctor X, inside the gloves that have touched the patient to perform the final operation.

We will use the following notation. There are m doctors, denoted $D_1, D_2, \ldots, D_m$, and n patients who are denoted $P_1, P_2, \ldots, P_n$. When a problem involves using k pairs of gloves, the pairs will be denoted $G_1, G_2, \ldots, G_k$. A pair of gloves that has been turned inside-out will be denoted $\overline{G}$. We list the doctors, gloves, and patients involved in a particular operation in that order. So the three operations in the solution to the original problem would be listed as:

1. $D_1\, G_1\, G_2\, P_1$
2. $D_2\, G_2\, P_1$
3. $D_3\, \overline{G_1}\, G_2\, P_1$

A variation of the problem has two doctors and two patients. Even though there are now *four* operations taking place, only two pairs of gloves are needed, and they do not even have to be turned inside-out! Doctor X wears both pairs 1 and 2 of gloves and operates on A. The outer pair 2 which have been contaminated on the outside by A are worn by themselves by Doctor Y to operate on A. Meanwhile, Doctor X operates on B wearing just pair 1 (whose outsides are still clean), touching B only with that clear side. Finally, Doctor Y puts the comtaminated-by-A outsides of pair 2 into the comtaminated-by-X insides of pair 1 to operate on B—who is still only being touched by the outsides of pair 1.

You should make sure you understand the notation of this problem by notating the solution just given to the two-doctor-two-patient problem; an answer will be given at the end of this reading section.

In the problems that follow, a *protocol* is a list of operations, in the order they are to be performed, showing which doctor, which patient, and which pair(s) of gloves are used, noting if any of the pairs have been turned inside-out. You should use the doctor-patient-glove notation as outlined above when needed. You may also use other standard mathematical notation, including the ceiling and floor functions, $\lceil x \rceil$ and $\lfloor x \rfloor$, as well as functions that choose the minimum or maximum of their inputs, $\min(x_1, x_2, \ldots, x_n)$ and $\max(x_1, x_2, \ldots, x_n)$.

You may invent other notations if convenient. If you do so you must either: 1) define the notation in each problem where it is used, or 2) include a separate sheet at the front of your solution packet where you gather all notations used and define them for the entire packet. This page should not include the solutions to any problems—notation definitions only!

For reasons that will become clear, a pair of gloves cannot be separated into two individual gloves. That is, both gloves from a pair must be worn for an operation, and if a doctor is wearing multiple pairs, each layer must consist of both gloves from the same pair. Furthermore, either both gloves are turned inside out or neither are. This justifies the notation using a single letter to represent a pair of gloves, rather than having to keep track of the left and right gloves separately.

The following is a fairly efficient protocol for $n = 2x$ patients and $m = 3y$ doctors, that uses exactly $x + 2y + 1$ pairs of gloves. Make sure you understand how this protocol works, as it is a model for some of the protocols you will need to construct in the problems.

For convenience, the gloves will be denoted $G_1, \ldots, G_x$, $H_1, \ldots, H_y$, $I_1, \ldots, I_y$, and J. Start by dividing the patients into pairs P_1 and Q_1, P_2 and Q_2, …, P_x and Q_x. Divide the doctors into trios, A_1-B_1-C_1, A_2-B_2-C_2, …, A_y-B_y-C_y.

Step 1: Give the pair G_i of gloves to patient P_i for i from 1 to x. Give pair H_j to doctor A_j, and pair I_k to doctor B_k for $1 \leq j, k \leq y$. Now each A- and B-doctor can operate on each P-patient via the operations $A_j\, H_j\, G_i\, P_i$ and $B_k\, I_k\, G_i\, P_i$. The insides of all H- and I-gloves are now contaminated and can only be touched by the doctor who wore them for this step. The outsides of each G-pair are contaminated by one patient who is the only one who can use that side from now on. The insides of the G-gloves and the outsides of the H- and I-gloves are still clean, though, and can be used by anyone.

Step 2: The I-gloves are turned inside-out and given to the C-doctors. The C-doctors now operate on the P-patients, using pair J to protect the inside of the G-gloves. That is, perform the operations $C_k\, \overline{I_k}\, J\, G_i\, P_i$ for all i and k. While performing these operations, the inside of glove J becomes contaminated by touching many different contaminated gloves, so that side as well as the (now) outsides of the I_k which have touched the inside of J can't ever touch any person. Meanwhile, the outside of the H-gloves and the inside of the G-gloves are still clean, and the P-patients have been operated upon by all the doctors.

Step 3: Since the P-patients are done and no longer need their gloves, turn them inside out and give them to the Q-patients. The C-doctors can now operate on all these patients, via the operations $C_k\, \overline{I_k}\, \overline{G_i}\, Q_i$. The C-doctors are now finished operating.

Step 4: Notice that the outsides of the H-gloves are still clean. Protect them with the clean outside of J, by performing all the operations $A_j\, H_j\, \overline{J}\, \overline{G_i}\, Q_i$.

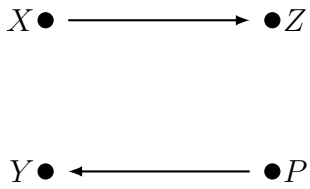
Step 5: Since the outsides of the H-gloves are clean, turn them inside-out and give them to the B-doctors. Perform all the operations $B_k\, \overline{H_k}\, \overline{G_i}\, Q_i$.

Notice that one side of J remains clean even though the other side has been contaminated by many different patients and doctors. Notice also we could even have started with one side of J already contaminated and it would have made no difference.

A protocol determines a diagram as follows. A diagram consists of points and arrows that connect the points (the arrows need not be drawn as straight lines). The points correspond to all the people, doctors and patients, involved in the protocol. The arrows correspond to gloves, and connect the first person who contaminates one side of the gloves with the person who first contaminates the other side.

More specifically, the first time a pair of gloves is contaminated (which may be by direct contact with a person, or by being placed in contact with an already-contaminated pair of gloves), an arrow starts at the point representing that person. If the contamination is with several people simultaneously (for example, the gloves are placed inside another pair which have already had several other contaminated pairs placed inside of them), then one of those people is chosen at random to be the beginning of the arrow. When the second side of the gloves become contaminated, the arrow ends at the point representing the person by whom it was contaminated. Again, if several contaminations occur simultaneously, any one of the people may be chosen. If the second side never becomes contaminated (like pair J above) then the arrow returns to the person with whom it started. Finally, if both sides of the gloves are contaminated for the first time simultaneously, the direction of the arrow can be chosen at random.

The diagram determined by the three-doctor-one-patient protocol is shown below:



Be sure you can draw diagrams corresponding to protocols. The diagram for the two-doctor-two-patient protocol is given at the end of this material.

You may use the language of directed graphs when discussing diagrams, if you feel comfortable doing so.

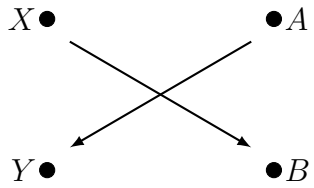
One final concept needs introduction. For a given protocol, its *makespan* is the amount of time it takes to complete it, allowing operations to take place simultaneously if they do not involve any of the same doctors, patients, or pairs of gloves. For the sake of convenience, assume all operations take exactly one hour and no time is needed between operations for patients, doctors, or gloves. The makespan of the protocol that solves the original problem is 3 hours, since each of three doctors operates in succession on the patient. The makespan

of the 2-doctor-2-patient problem is also 3 hours. This is because the middle two operations can be performed simultaneously. The makespan of the protocol for $2x$ patients and $3y$ doctors depends on x and y and is quite long, as several steps require pair J to be used in each operation!

The notation for the two-doctor-two-patient solution given in the text is:

1. $D_1\, G_1\, G_2\, P_1$
2. $D_2\, G_2\, P_1$
3. $D_1\, G_1\, P_2$
4. $D_2\, G_2\, G_1\, P_2$

This is one way to draw the corresponding diagram.



Instructions for students

This is the second round of the 2021–2022 ARML Power Contest and should take place between Saturday, February 19 and Sunday, March 6.

You will have 45 minutes to complete this set of problems. During this time you are encouraged to work together, communicating and sharing ideas. It is important that everyone understands the problem before splitting up into smaller groups. There is a considerable amount of background material which you should either have already been given to read, or that your proctor will give you and allow you to read and discuss for a while before beginning the actual contest problems. Make sure each person understands how the various notations, procedures, and computations work and the meanings of the vocabulary terms introduced in that material before beginning the actual contest.

For teams working face-to-face, language translators may be used on this contest, but no other electronic devices are allowed. This includes, but is not limited to, calculators, computers, tablets, and smartphones. Teams that are working together online are, obviously, going to use electronic communication; these teams are on the honor system to only use their electronic communication equipment to work together and not try to find information about the problem online; keep in mind that many online sources about this problem contain incorrect information so looking there will not help!

The topic of the spring contest this year is inspired by a puzzle by Martin Gardner. It involves finding the minimum amount of protective equipment needed for a group of patients and doctors to safely interact with each other.

Please pay careful attention to the directions in each contest question, as that may save you quite a bit of writing! Remember that this is a contest of mathematical writing, and many problems require mathematical justification or proof. Be clear and concise. You may refer to the result of an earlier problem in your work—even if you didn't solve that problem. You may also refer to your work from an earlier problem. You may not refer to later problems, however, even if it does not create circular arguments. Please keep in mind that the problems are *not* ordered in increasing order of difficulty. You may be able to solve some later problems even if you cannot solve some of the earlier ones.

Use dark pencil or ink and please be legible. *Write on only one side of each answer sheet.* You may submit several answers on the same answer sheet, but be sure to submit only one solution for each problem! Answers on the backs of sheets that are not seen by the graders will receive no credit. If multiple answers to a single problem are found, all will be graded and the team will earn the *lowest* score for any of their solutions. At the conclusion of the contest, submit your solutions *in order* to your coach. Problems that are out of order (except when multiple solutions are written on the same page) have sometimes been overlooked by graders, and there is unfortunately no way to give credit retroactively if this happens.

The contest has a total of 40 points possible. You will be given a copy of the scoring sheet; the point value of each problem is also listed on the problem sheets.

You are encouraged to include any comments or concerns about the problem on the comment sheet.

Contests will be mailed in or submitted electronically, and scores will be posted online as soon as possible. The solutions will also be posted for coaches to download.

Good luck, and enjoy!

2021–2022 ARML Power Contest
Round 2
Grading Sheet

Team ID: _____

1. ____ [2]
2. ____ [2]
3. ____ [2]
4. ____ [2]
5a. ____ [2]
5b. ____ [2]
6. ____ [2]
7. ____ [3]
8. ____ [2]
9. ____ [1]
10. ____ [2]
11. ____ [1]
12. ____ [2]
13. ____ [2]
13. ____ [2]
14a. ____ [3]
14b. ____ [1]
15. ____ [2]
16. ____ [3]

Total score: _____

We have complied with the rules of this contest. _____
(Coach's signature)

_____
(Date)

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Round 2
Answer Sheet

Page ____ of ____. Problem(s) _____.

Team ID: _____

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2021–2022 ARML Power Contest
Round 2
Comment Sheet